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Periods of ordinary abelian varieties in characteristic p

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Abstract. For ordinary abelian varieties in characteristic $p > 0$, we define an analogue of the period lattice and of the parametrization by $\mathbb{C}^g$ and give some applications.
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Introduction

If A is an abelian variety defined over the complex numbers $\mathbb{C}$, then there exists a lattice $\Lambda \subset \mathbb{C}^g$, $g = \dim A$, such that $A(\mathbb{C}) = \mathbb{C}^g/\Lambda$. This lattice is called the period lattice because functions on A will be periodic functions on $\mathbb{C}^g$ with periods in Λ. In this note we give an analogue in characteristic p for the period lattice Λ and for the parametrization $A(\mathbb{C}) = \mathbb{C}^g/\Lambda$. For now on, all our fields are assumed to be of characteristic $p > 0$.

Notation: For an abelian group H we define $\hat{H} = \varprojlim H/p^n H$. Then $\hat{H}$ is a $\mathbb{Z}_p$-module.

Let A be an ordinary abelian variety over a field K of characteristic $p > 0$ and let K_s be the separable closure of K and $G = \mathrm{Gal}(K_s/K)$. Let $A^{(p^n)}$ be the image of A under the n-th power of the Frobenius map F^n and $V_n \colon A^{(p^n)} \to A$ the dual isogeny, the n-th order Verschiebung, which is separable since A is ordinary. Then $\ker V_n$ is the p^n torsion of $A^{(p^n)}$. We define the period lattice by $\Lambda = \varprojlim \ker V_n$. This definition is not new, it corresponds to the Serre-Tate parameters (see e.g. [**K**]), however it is usually only considered when the ground field is a local field. See [**K**] also for the relationship between the Serre-Tate parameters and moduli.

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The generalization of the analytic parameterization of an abelian variety is given by the following:

Theorem 1. *There exists maps that make the following an exact sequence of G-modules:*

$$\Lambda \to \widehat{K_s^*} \otimes \Lambda^{\otimes(-1)} \to \widehat{A(K_s)} \to 0.$$

A few comments are in order. As a $\mathbb{Z}_p$-module, $\widehat{K_s^*} \otimes \Lambda^{\otimes(-1)}$ is isomorphic to $(\widehat{K_s^*})^g$, $g = \dim A$, but they are different as G-modules. Secondly, the analogy with the analytic parameterization is more evident after composing it with the exponential map. This construction also generalizes the Tate parametrization of elliptic curves: If K_v is a local field and E/K_v is an elliptic curve with split multiplicative reduction then $\exists q \in K_v$ such that $0 \to q^{\mathbb{Z}} \to K_v^* \to E(K_v) \to 0$ ([**S**], Ch. V). It is easy to show that Theorem 1 follows from the Tate parametrization in this case (see [**V1**], lemma 2). This generalizes to Mumford's parametrization of abelian varieties with completely multiplicative reduction [**Mu**].

Proof. We have the exact sequence of group schemes

$$0 \to \ker F^n \to \ker[p^n] \to \ker V_n \to 0.$$

Taking flat cohomology yields

$$\ker V_n(K_s) \to H^1(K_s, \ker F^n) \to H^1(K_s, \ker[p^n]) \to 0.$$

We will analyze the terms of this sequence and show that it gives the theorem by passing to the inverse limit. First of all, $\varprojlim \ker V_n(K_s) = \Lambda$, by definition. On the other hand, $H^1(K_s, \ker[p^n]) = A(K_s)/p^n A(K_s)$, which follows from the exact sequence $0 \to \ker[p^n] \to A \to A \to 0$ and the fact that $H^1(K_s, A) = 0$, since K_s is separably closed. As for the middle term, Cartier duality gives

$$H^1(K_s, \ker F^n) = H^1(K_s, \mu_{p^n}) \otimes \ker V_n^{\otimes -1} = K_s^*/(K_s^*)^{p^n} \otimes \ker V_n^{\otimes -1}.$$

Putting these together and passing to the inverse limit yields the theorem.

Corollary. *If K is a global field, E/K an elliptic curve and v a place of K where E has bad reduction, then q is transcendental over K and so is any $u \in K_v^*$ which maps to a point of infinite order in $E(K)$.*

This corollary is proved in detail in [**V**]. The proof consists in comparing the Tate parametrization and the parametrization given by Theorem 1 and using a theorem of Igusa which guarantees that the action of G on Λ is not through a finite quotient. The transcendence of q is the characteristic p analogue of the recent result of Barré-Sirieix *et al.* [**B**]. It would be nice to generalize the corollary to higher dimensional abelian varieties. This would require understanding the action of G on Λ. The result follows, for example if G acts via the full general linear group, which is the generic case by [**FC**], Prop. V.7.1.

Another application of theorem 1 is to local duality. It is a classical result of Tate (up to the p-part in characteristic p, which is due to Milne) that, if K is a local field with finite residue field, then $A(K)$ and $H^1(G, A(K_s))$ are Pontrjagin duals. There is a conjecture of Milne ([**M**], III, Conjecture 10.7) which generalizes the local duality to case of algebraically closed residue field. This conjecture is known in the case of good reduction (Bester, see [**M**]) and for elliptic curves with split multiplicative reduction (Shatz, see [**M**]). We extend these results a bit in the following proposition, and we believe its proof may be extended to give further results along these lines.

Proposition. *Let K be a local field whose residue field is the algebraic closure of a finite field of characteristic $p > 0$ and A/K an abelian variety whose reduction is a semi-abelian variety with ordinary abelian quotient. Assume also that $A[p] \cap A(K_s) = 0$, then $T_p(H^1(G, A(K_s))$ is isomorphic to $H^1(G, \Lambda)$.*

Proof. Consider the exact sequence of G-modules

$$0 \to K_s^* \xrightarrow{p^n} K_s^* \to K_s^*/(K_s^*)^{p^n} \to 0.$$

Under the hypotheses of the theorem, $H^1(G, K_s^*) = H^2(G, K_s^*) = 0$, so the Galois cohomology sequence of the above exact sequence yields $(\widehat{K_s^*})^G = \widehat{K^*}$ and $H^1(G, \widehat{K_s^*}) = 0$.

Now consider the exact sequence of G-modules

$$0 \to A(K_s) \xrightarrow{p^n} A(K_s) \to A(K_s)/p^n A(K_s) \to 0.$$

Taking Galois cohomology yields

$$0 \to \widehat{A(K)} \to \widehat{A(K_s)}^G \to T_p(H^1(G,A)) \to 0.$$

By our hypothesis on the reduction type of A we obtain that V_n is étale on the special fibre as well as on A, thus $\Lambda = \mathbb{Z}_p^g$ with the trivial action of G. It also follows that V_n is an isomorphism on the formal group of A. Thus, given $P \in A(K)$ in the formal group, we can find $Q \in A^{(p^n)}(K)$, $V_n(Q) = P$ and we can then map Q to $H^1(K, \ker F^n)$ using the coboundary map of the flat cohomology sequence coming from the exact sequence $0 \to \ker F^n \to A \to A^{(p^n)} \to 0$. This gives us an inverse, in the formal group, to the map $(\widehat{K^*})^g = \varprojlim H^1(K, \ker F^n) \to \widehat{A(K)}$ which comes from theorem 1. It follows that $\widehat{A(K)} = (\widehat{K^*})^g/\Lambda$.

We are now ready to take Galois cohomology of the exact sequence of theorem 1. Note that under our present assumptions this sequence is exact on the left also. We get

$$0 \to \Lambda \to (\widehat{K^*})^g \to \widehat{A(K_s)}^G \to H^1(G,\Lambda) \to 0.$$

Since $(\widehat{K^*})^g$ surjects onto $\widehat{A(K)}$, we obtain that

$$T_p(H^1(G,A)) = \widehat{A(K_s)}^G / \widehat{A(K)} = H^1(G,\Lambda).$$

We say that an ordinary abelian variety A is sufficiently general if $A[p^\infty] \cap A(K_s)$ is finite. It follows from the proof of theorem 1, that A is sufficiently general if and only if the map $\Lambda \to \widehat{K_s^*} \otimes \Lambda^{\otimes(-1)}$ is injective. In [**V2**] a sufficient condition for A to be sufficiently general is given which justifies the name "sufficiently general". The following theorem studies the action of the endomorphisms of A on Λ and produces a best possible result under the hypotheses, showing that Λ behaves like the period lattice in this case and also like the ℓ-adic representation.

Theorem 2. *The natural map* $\mathrm{End}(A) \otimes \mathbb{Z}_p \to \mathrm{End}(\Lambda)$ *is injective if A is sufficiently general.*

Proof. It suffices to show, by standard arguments, that if $\phi \in \mathrm{End}(A)$ acts trivially (via $\phi^{(p^n)}$) on $\ker V_n$ for n large, then ϕ factors through $[p]: A \to A$. Let $\check{A}$ be the dual abelian variety and fix a polarization $\alpha: A \to \check{A}$, defined over K. We have a dual map $\check{\phi}: \check{A} \to \check{A}$ and $\check{\phi}$ kills the Cartier dual of $\ker V_n$ which is $\ker F^n$ on $\check{A}$. We can thus factor $\check{\phi} = \psi \circ F^n$, $\psi: \check{A}^{(p^n)} \to \check{A}$. We are done if $\check{\phi}$ kills $\ker[p]$. But, if that is not the case there exists a cyclic subgroup H of $\check{A}^{(p^n)}$ of order p^n on which ψ is injective. This subgroup will, moreover, be defined over K_s. Thus, $\alpha(\psi(H))$ is a large subgroup of A of p-power order defined over K_s, which will be a contradiction for n sufficiently large.

One may conjecture, transposing a similar conjecture of Tate, that $\mathrm{End}(A) \otimes \mathbb{Z}_p$ is isomorphic to $\mathrm{End}_G(\Lambda)$, if A is defined over a global field K with absolute Galois group G and A is sufficiently general. This is trivial if A is an elliptic curve, since both groups are isomorphic to $\mathbb{Z}_p$ under the hypotheses. The first non-trivial case is when A is a product of two elliptic curves and in this case the conjecture is true, being essentially equivalent to Keating's characterization of the Igusa tower [**Ke**].

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